Simultaneous Solution of the Boundary Layer and Freestream with Separated Flow

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A general procedure for calculating the boundary layer simultaneously with the outer, inviscid flow is described. Integral equations for the boundary layer and finite-difference equations for the inviscid flow at a given longitudinal position form a linear set with a tridiagonal coefficient matrix. The set is solved simultaneously at each position, beginning at the upstream boundary and iterating over the flowfield (successive line relaxation). Convergence of the procedure with separated flow is demonstrated by a numerical example.

Nomenclature

a	= coefficients appearing in Eqs. (4) and (5)
b	= coefficients appearing in Eqs. (4) and (5)
C_f	= skin friction coefficient, $\tau_0/\frac{1}{2}\rho U^2$
f	= coefficients appearing in Eqs. (6-9)
g	= coefficients appearing in Eqs. (6-9)
$2h$	= channel height at entrance region
H_{12}	$=\delta_1/\delta_2$
H_{32}	$=\delta_3/\delta_2$
R	$=\Delta y/\Delta x$
s	= coordinate aligned with solid boundary
u	= velocity in boundary layer
U	= freestream velocity at edge of boundary layer
U_I	= inlet velocity
x	= longitudinal coordinate
Δx	= grid spacing in x direction
y	= transverse coordinate
Δy	= grid spacing in y direction
Δy_I	= grid spacing defined in Fig. 3
β	= velocity profile shape factor
δ	= boundary-layer thickness parameter
δ_1	= displacement thickness
δ_2	= momentum thickness
δ_3	= energy thickness
η	$=y/\delta$
ν	= kinematic viscosity
ρ	= density
τ_0	= wall shear stress
ψ	= stream function

I. Introduction

THE failure of early attempts to analytically predict the drag forces on submerged bodies with potential flow theory is known as D'Alembert's paradox. The addition of viscous terms to the equations of motion was of little help, however, because of the difficulties in obtaining solutions, except for very limited cases. Thus Prandtl's boundary-layer theory, with the simpler viscous equations and the implication that it could be solved separately from the inviscid freestream, had great promise.

The solution of the boundary-layer equations for a given pressure distribution has received a great deal of attention in this century. With modern computers, it is possible to obtain a very accurate solution for two-dimensional, laminar flow up to the point of separation. With sufficient boundary conditions, the three-dimensional boundary-layer equations can also be solved numerically for laminar flow. Transition and turbulence still require some empiricism, but acceptable solutions can usually be obtained.

For a fixed pressure distribution, the boundary-layer equations become singular at the separation point (see for example, Ref. 1). With backflow near the wall past separation, the equations become unstable in the downstream direction. Furthermore, when separation is involved, there is a strong interaction between the freestream and the boundary layer, and the two cannot be determined separately.

Because of these difficulties and the increased viscous-layer thickness with separation, it is usually assumed that the boundary-layer equations do not apply. Thus, with limited exceptions, flows involving separation from smooth walls have not been treated analytically. The extremely important phenomenon of stall must still be dealt with by experimentation and a qualitative understanding of the flow. Kline and Dean[2] list flow separation as one of the most important problems yet to be solved in fluid mechanics. In most applications, the problem is greatly complicated by secondary flow, turbulence, and unsteady effects in addition to the numerical difficulties.[3]

Because of the problems associated with the boundary-layer equations at separation and the availability of digital computers, numerical solutions of the Navier-Stokes equations have received much attention in recent years. Interesting results have been obtained for laminar flow with relatively simple geometries (e.g., Ref. 4). Numerical solutions for turbulent flow have also been obtained,[5] but involve some empiricism for the Reynolds stresses. While this approach admittedly has great promise, it is clearly not the best for all separated flow problems. Even if computers were large and fast enough to handle complex geometries such as multistage, axial-flow compressors, it would be extremely difficult to gain an understanding of the physical significance of the many variables by this approach.

Although the boundary layer is not necessarily thin with separation, many real flows of interest can be divided into one

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region where viscous forces are important and another region that is essentially inviscid. For a finite Reynolds number, the velocity derivatives do not become infinite at separation,[1] but there is some region where the boundary-layer approximations are not strictly valid. If this region is sufficiently small, as, for example, the leading edge of a flat plate, it is quite probable that the approximations do not appreciably affect the overall flow. The error involved is one of degree, as it is for any boundary-layer approximation with a finite Reynolds number.

Even if the boundary-layer approximations are accepted for separated flow, a general solution procedure is still needed. Because of the singular behavior, the downstream instability, and the strong interaction between the boundary layer and the freestream, a straightforward iteration is not possible.[6] A simultaneous solution, however, eliminates these difficulties. Abbott et al.[7] and Lees and Reeves[8] obtained simultaneous solutions for supersonic flow, where the freestream is governed by a hyperbolic equation, by relating the turning angle to the boundary-layer growth. Because of the elliptic nature of the freestream equation for subsonic flow, the solution is much more difficult. Moses and Chappell[9] obtained a simultaneous solution by assuming a one-dimensional pressure distribution, which changes the nature of the freestream equation, but this method is not general. Bower[10] used a similar approach for subsonic, compressible flow. Ghose and Kline[11] also used a one-dimensional pressure assumption, with improved turbulence correlations, and then extended the procedure to the two-dimensional case by an iterative scheme.

The purpose of this paper is to describe a general procedure for simultaneously calculating the boundary layer and the freestream. The method is quite general, but is described here in its simplest form—two-dimensional, laminar, and incompressible flow. A numerical example is included to demonstrate the convergence of the procedure with separated flow.

II. Description of General Procedure

Consider a flowfield that can be divided into an essentially inviscid region and boundary layers near solid surfaces (Fig. 1). It is desired, in general, to find a solution for the flowfield over some region, with either the boundaries prescribed (direct problem) or with a flow property such as the surface pressure prescribed (indirect problem). An approximate, macroscopic solution for the forces on the solid boundaries and the total pressure losses is usually satisfactory.

For subsonic flow, boundary conditions must be prescribed completely around the area of interest for the problem to be mathematically well-posed. These conditions usually include the upstream flow, no-slip along solid walls, uniform flow at large distances for external flow, and the downstream flow. In practical applications, the boundary conditions are not known exactly, especially with respect to the downstream flow. It is usually assumed that the calculation can be carried sufficiently far downstream so that conditions there will not affect the region of interest. Furthermore, a steady state is often assumed, and this is not mathematically sufficient in all cases. (Consider bistable flow in a diffusing passage, for example, or turbulent flow.) Thus, any solution to a real flow problem is likely to involve some degree of approximation.

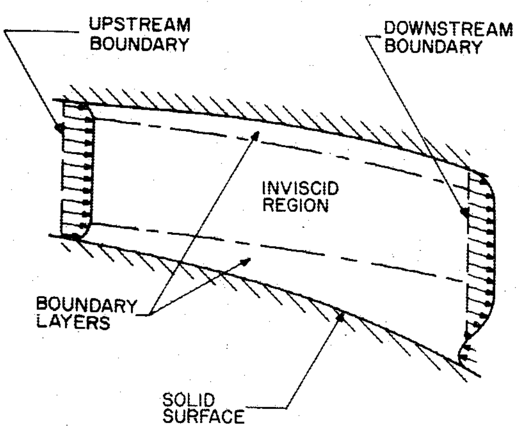

Fig. 1 Two-dimensional flowfield.

A. Inviscid Flow

The theory of inviscid flow has reached a very high state of development in both analytical and numerical techniques. With digital computers, finite-difference methods are available for treating rotational or irrotational and compressible or incompressible flows for arbitrary geometrical shapes.

Since the objective here is to present a basic method of treating separated flow, the inviscid region will be assumed incompressible and irrotational. (This assumption is not necessary, but greatly simplifies the discussion.) If it is further assumed that the flow is two-dimensional, Laplace's equation for the stream function can be applied:

$$\nabla^2\psi = 0 \tag{1}$$

The boundary conditions for Eq. (1) are the values of the stream function or its derivative on all boundaries (elliptic).

Equation (1) can be solved by any one of a number of different numerical approaches which are stable and convergent.[4] The method employed here is successive line relaxation (SLR). For simplicity, it is discussed without overrelaxation (relaxation factor of unity). In this approach the unknowns at each value of i (see Fig. 2) are determined simultaneously, with values at $i-1$ from the preceding calculation and $i+1$ from the previous iteration. For any value of i, the equations can be written

$$\alpha_{i,j-1}\psi_{i,j-1} + \alpha_{i,j}\psi_{i,j} + \alpha_{i,j+1}\psi_{i,j+1} = \gamma_{i,j} \tag{2}$$

The coefficients $\alpha_{i,j}$ and $\gamma_{i,j}$ are functions of the grid spacing. Since the matrix is tridiagonal, the set can be solved quickly with a simple algorithm.[12]

The solution is completed by an iteration over i until a satisfactory convergence, usually determined by a maximum permitted change in ψ, is achieved.

B. Boundary Layers

Over seventy years of research have resulted in a large number of methods for calculating boundary-layer development. For laminar flow, numerical methods are relatively straightforward and quite accurate when the pressure is known. However, the pressure is not known exactly, and the boundary-layer development is extremely sensitive to the pressure gradient near separation. Thus, a highly accurate solution to the boundary-layer equations is somewhat academic when separation is involved.

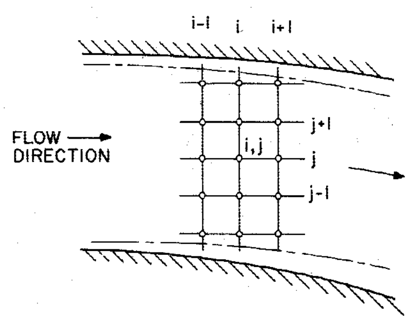

Fig. 2 Grid system for inviscid flow region.

The general procedure described here employs a two-parameter integral method for the boundary layer. (A numerical method or a larger number of parameters could be used, but at the cost of considerable complexity.)

The velocity profile is assumed in parametric form,

$$u/U=f(y/\delta,\beta) \tag{3}$$

where δ is a thickness parameter and β is a profile shape parameter. Equation (3) can be an explicit relation, such as a polynomial, or in functional form, such as solutions to the Falkner-Skan equation.

Ordinary differential equations for the unknown parameters are derived by integrating the boundary-layer equation and/or its moments in u or y across the boundary layer, or by integrating across strips within the boundary layer. Most methods for laminar or turbulent flow result in equations of the form,

$$a_{11}\mathrm{d}\delta+a_{12}\mathrm{d}\beta=b_{11}\mathrm{d}U+b_{12}\mathrm{d}s \tag{4}$$

and

$$a_{21}\mathrm{d}\delta+a_{22}\mathrm{d}\beta=b_{21}\mathrm{d}U+b_{22}\mathrm{d}s \tag{5}$$

For attached flow with a known freestream velocity, these equations can be solved numerically with a stepwise procedure *in the downstream direction.* For most velocity profile assumptions, the equations are singular at separation (the determinant of the coefficient matrix a_{ij} is zero) and a solution only exists for a specific pressure gradient. Thus, the pressure cannot be previously specified with sufficient accuracy to avoid infinite derivatives at this point. Past the point of separation (or in the upstream direction) the equations are unstable. This fact can be easily demonstrated by considering the pressure gradient for which $\mathrm{d}\beta=0$ for a given value of β (equilibrium) and noting that a small change in β will cause the solution to move away from equilibrium.

In the singular behavior at separation and the instability downstream, the approximate integral equations for the boundary layer are similar to the partial differential equations. The instability is typical of parabolic equations with a dissipative mechanism, and is related to the second law of thermodynamics. A solution carried out under these conditions will not, in general, be reliable.

C. Simultaneous Solution of the Boundary Layer and Freestream

The numerical difficulties associated with separation can be avoided if the freestream and boundary-layer equations can be solved simultaneously. Because of the elliptic nature of the freestream equation (for subsonic flow), a simultaneous solution is not straightforward. The procedure suggested here is to solve the equations simultaneously at each longitudinal (i) position, beginning at the upstream boundary and iterating over the flow field. By arranging the set of equations so that the coefficient matrix is tridiagonal, the computational time can be considerably reduced.

The boundary-layer equations (in integral form) are written with the displacement thickness as one parameter and a shape factor as the other. The freestream velocity at the edge of the boundary layer is treated as an unknown and eliminated from one equation to make the coefficient matrix tridiagonal for the complete set of equations. Thus, the boundary-layer equations can be written,

$$f_{11}\mathrm{d}\delta_1+f_{12}\mathrm{d}\beta+f_{13}\mathrm{d}U=g_1\mathrm{d}s \tag{6}$$

and

$$f_{21}\mathrm{d}\delta_1+f_{22}\mathrm{d}\beta=g_2\mathrm{d}s \tag{7}$$

Equations (6) and (7) are written in finite-difference form, with upstream differencing.

$$f_{11}\delta_{1_i}+f_{12}\beta_i+f_{13}U_i=g_1\Delta s+f_{11}\delta_{1_{i-1}}+f_{12}\beta_{i-1}+f_{13}U_{i-1} \tag{8}$$

and

$$f_{21}\delta_{1_i}+f_{22}\beta_i=g_2\Delta s+f_{21}\delta_{1_{i-1}}+f_{22}\beta_{i-1} \tag{9}$$

where the coefficients f and g are functions of $\delta_{1_{i-1}}$, β_{i-1} and U_{i-1}. These functions depend on the method of derivation and velocity profile assumption. (The effect of this assumption is discussed in the next section.)

For incompressible flow, the potential flow equations can be written as Eq. (10) for all but the point just outside the boundary layer.

$$\psi_{i,j-1}-(2+2R^2)\psi_{i,j}+\psi_{i,j+1}=-R^2(\psi_{i-1,j}+\psi_{i+1,j}) \tag{10}$$

For the grid point just outside the boundary layer, the unkown $\psi_{i,1}$ can be eliminated with the freestream velocity (see Fig. 3).

$$U_i=\frac{\psi_{i,2}-\psi_{i,1}}{\Delta y}\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)_i \tag{11}$$

Equation (11) assumes that the slope of the displacement thickness is approximately that of the wall, which is consistent with the boundary-layer assumptions.

With Eq. (11), the potential flow equation for this point is

$$-\left[\Delta y\Big/\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)_i\right]U_i-(1+2R^2)\psi_{i,2}+\psi_{i,3}$$
$$=-R^2(\psi_{i-1,2}+\psi_{i+1,2}) \tag{12}$$

The set is completed with a relation between the freestream velocity and the stream function on the displaced boundary, which is assumed constant (see Fig. 3).

$$U_i=\frac{\psi_{i,2}-\psi_{\delta 1}}{\Delta y_1-\delta_{1_i}}\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)_i \tag{13}$$

Equation (13) is differentiated with upstream finite differences.

$$\frac{U_{i-1}}{(\mathrm{d}s/\mathrm{d}x)_{i-1}}\delta_{1_i}-\frac{(\Delta y_{1_{i-1}}-\delta_{1_{i-1}})}{(\mathrm{d}s/\mathrm{d}x)_{i-1}}U_i+\psi_{i,2}=\psi_{i-1,2}$$
$$-\left\{-\Delta y_{1_i}+\Delta y_{1_{i-1}}\left[1+\frac{(\mathrm{d}s/\mathrm{d}x)_i}{(\mathrm{d}s/\mathrm{d}x)_{i-1}}\right]\right.$$
$$\left.-\delta_{1_{i-1}}\left[1+\frac{(\mathrm{d}s/\mathrm{d}x)_i}{(\mathrm{d}s/\mathrm{d}x)_{i-1}}\right]\right\}\frac{U_{i-1}}{(\mathrm{d}s/\mathrm{d}x)_{i-1}} \tag{14}$$

Equations (8-10, 12, and 14) form a linear set of simultaneous equations with a tridiagonal coefficient matrix.

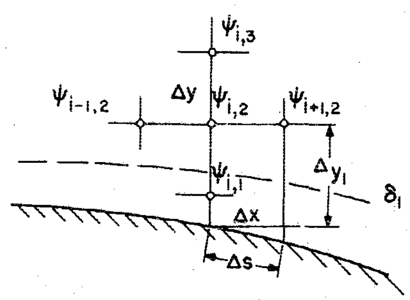

Fig. 3 Grid system at boundary layer.

III. Numerical Example

To demonstrate the convergence of the general procedure, the calculation of separated flow in a two-dimensional diffusing passage is described. Consideration here is limited to low Reynolds numbers or laminar boundary layers to avoid the additional difficulties associated with turbulent separation. To simplify the discussion, the flow is assumed symmetrical, so that only one boundary layer is considered.

The physical dimensions of the two-dimensional channel are shown in Fig. 4. The wall shape for the diffusing section, between the two constant-area sections, is a cubic curve, and the wall slope is continuous.

Upstream boundary conditions include a uniform freestream (linear variation of ψ) and a flat plate boundary layer with a specified displacement thickness ($\delta_l/2h = 0.015$). No slip is assumed along the solid wall, and the centerline is a streamline (constant ψ) from symmetry. [For the inviscid flow, a streamline (constant ψ) is assumed to follow the boundary-layer displacement.] At the downstream boundary, the inviscid flow is assumed uniform (linear ψ). Since the boundary layer equation does not have a downstream boundary condition (parabolic), the entire solution must proceed from upstream to this point on each iteration. Values are not calculated *on* the downstream boundary, however, so the displacement thickness is approximated by the value calculated for the location just upstream.

The boundary-layer calculations are made with a method similar to that of Tani[13] where the velocity profiles are assumed in parametric form.

$$u/U = (2\eta - \eta^2) + \beta(\eta - 2\eta^2 + \eta^3) \tag{15}$$

(In addition to the above cubic profile, the quartic as used by Tani[13] and the Falkner-Skan profiles as used by Lees and Reeves[8] are included to determine the effect of the assumed profiles on the result.)

Ordinary differential equations are derived from the momentum integral equation and the velocity moment, or kinetic energy, integral equation.

$$\frac{d\delta_2}{dx} + (H_{12}+2)\frac{\delta_2}{U}\frac{dU}{dx} = \frac{C_f}{2} \tag{16}$$

$$\frac{d(U^3\delta_3)}{dx} = 2\nu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy \tag{17}$$

With the above profiles, Eqs. (16) and (17) can be written in the form of Eqs. (8) and (9). The complete set of equations used in the numerical example are given in the Appendix.

The calculations were started with initial assumptions that included a constant boundary-layer thickness and stream functions based on one-dimensional flow. The equations for each longitudinal position were solved simultaneously, starting at the upstream boundary and proceeding downstream. The procedure was repeated until convergence, determined by the maximum change at any point, was achieved.

The results of the calculations for the two-dimensional diffusing passage are shown in Figs. 5 and 6. In Fig. 5 the vertical scale is increased to show the development of the boundary layer through separation and reattachment. The freestream velocity, boundary-layer displacement thickness, and wall shear stress are shown in Fig. 6 for the three profile assumptions.

To demonstrate the difficulties associated with attempting to calculate the boundary layer through separation with a fixed pressure distribution, Eqs. (8) and (9) were solved independently. The pressure distribution used for this latter calculation was that determined from the simultaneous solution. Figure 7 shows these results with the boundary-layer calculation started before and after separation. A very small change (0.4%) in the upstream boundary-layer thickness, with a fixed pressure distribution, also gave widely different values.

IV. Conclusions

By calculating the freestream simultaneously with the boundary layer, the singularity at the separation point is avoided and the solution is stable in the separated region.

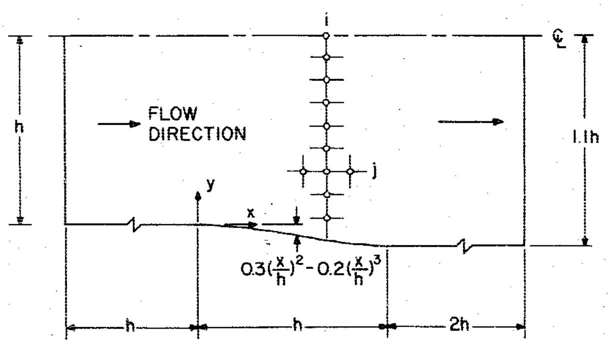

Fig. 4 Channel for numerical example.

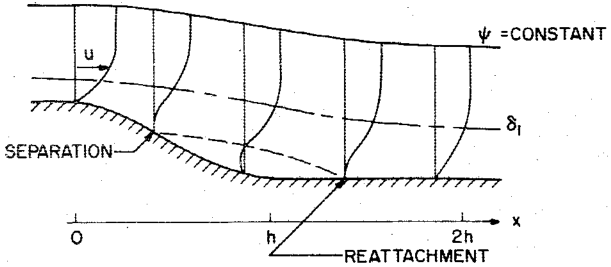

Fig. 5 Boundary-layer velocity profiles.

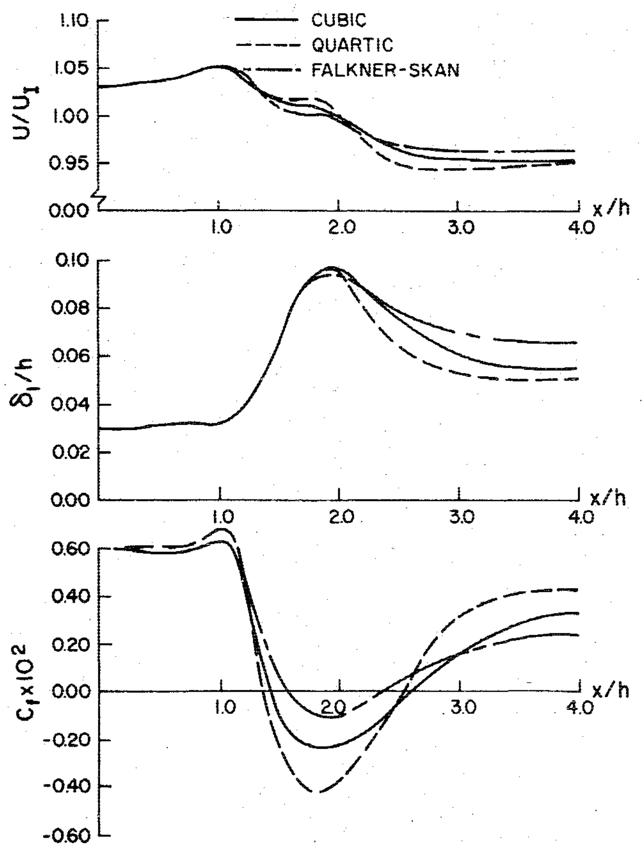

Fig. 6 Results of simultaneous boundary-layer and freestream calculations with different velocity profile assumptions.

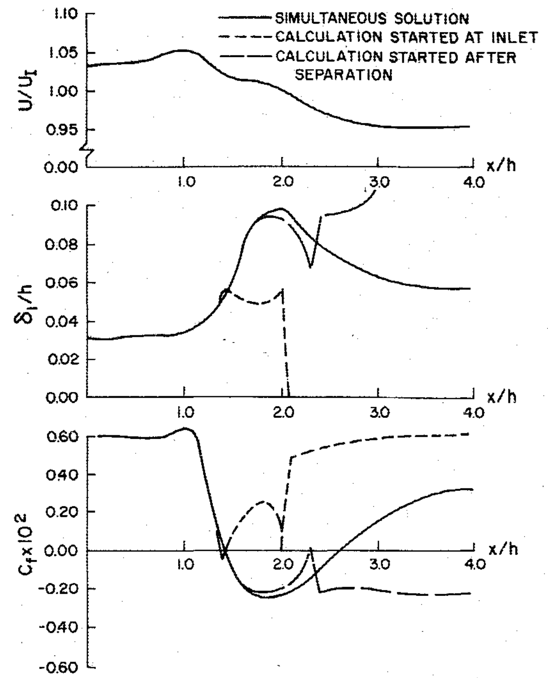

Fig. 7 Results of boundary-layer calculations with fixed pressure distribution.

With an elliptic freestream equation, an iteration over the flowfield is required, but convergence is quite rapid. The simultaneous solution is, in fact, only slightly more difficult than the inviscid solution alone. The method is also useful for flows that do not involve separation, especially if there is a strong interaction between the boundary layer and inviscid region. The procedure can be extended to more general cases, which include compressibility, turbulence, and outer flows with rotation.

The integral method for the boundary layer is, admittedly, approximate and depends on the profile assumption as well as the derivation of the ordinary differential equations. The overall flow is not extremely sensitive to these approximations, however, especially if the separated region is relatively small. Profiles that have a limited backflow velocity, such as solutions to the Falkner-Skan equations, are more realistic when the separated region is a large part of the total viscous layer. A more accurate procedure could be used for the boundary layer, but the calculations in the separated region would not be unconditionally stable, at least in the cases where the pressure is assumed constant across the boundary layer.

With a thick viscous region usually associated with separated flow, the degree of approximation in the boundary-layer approach is increased. However, the largest errors are near the points of separation and reattachment and do not greatly affect the overall flow, especially if there is a relatively large inviscid region. Furthermore, any solution to this problem is likely to be somewhat approximate.

The difficulty of attempting to calculate the boundary layer through separation *with a fixed pressure distribution*, which eliminates the possibility of a straightforward iteration with the inviscid flow, is clearly demonstrated. The most obvious difficulty is associated with the singularities at separation and reattachment. Even if the singularities are somehow circumvented, the parabolic boundary-layer equations are unstable in a direction opposite to that of the flow.[4] (This situation is similar to calculating the transient temperature in a body backwards in time.) The problem is not always obvious because reasonable answers can sometimes be obtained for short excursions into an unstable region. However, infinitesimally small errors can become greatly amplified and calculations of this type may produce serious errors.

Appendix: Equations Used in Numerical Example

For the numerical example, a uniform grid spacing was utilized in the x and y directions which yields $R = 1$.

Written in matrix format, the coupled equations describing the boundary layer and freestream take the following form:

$$\begin{bmatrix} B_1 & C_1 & & & & & \\ A_2 & B_2 & C_2 & & & & \\ & A_3 & B_3 & C_3 & & & \\ & & A_4 & B_4 & C_4 & & \\ & & & A_5 & B_5 & C_5 & \\ & & & & \vdots & & \\ & & & & & A_{k\max} & B_{k\max} \end{bmatrix}$$

$$\times \begin{bmatrix} \beta_i \\ \delta_{1_i} \\ U_i \\ \psi_{i,2} \\ \psi_{i,3} \\ \vdots \\ \psi_{i,j\max-1} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ \vdots \\ D_{k\max} \end{bmatrix}$$

where

$$B_1 = (a_{22}b_{11} - a_{12}b_{21})\frac{dH_{12}}{d\beta} \qquad C_1 = a_{21}b_{11} - a_{11}b_{21}$$

$$A_2 = a_{12}\frac{dH_{12}}{d\beta} \qquad B_2 = a_{11} \qquad C_2 = -b_{11}$$

$$A_3 = \frac{U_{i-1}}{(ds/dx)_{i-1}} \qquad B_3 = \frac{-(\Delta y_{1_{i-1}} - \delta_{1_{i-1}})}{(ds/dx)_{i-1}} \qquad C_3 = 1$$

$$A_4 = \frac{-\Delta y}{(ds/dx)_i} \qquad B_4 = -3 \qquad C_4 = 1$$

$$A_5 = 1 \qquad B_5 = -4 \qquad C_5 = 1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$A_{k\max} = 1 \qquad B_{k\max} = -4$$

and

$$D_1 = (b_{22}b_{11} - b_{12}b_{21})\Delta s + (a_{22}b_{11} - a_{12}b_{21})\frac{dH_{12}}{d\beta}\beta_{i-1} + (a_{21}b_{11} - a_{11}b_{21})\delta_{1_{i-1}}$$

$$D_2 = b_{12}\Delta s + a_{12}\frac{dH_{12}}{d\beta}\beta_{i-1} + a_{11}\delta_{1_{i-1}} - b_{11}U_{i-1}$$

$$D_3 = \psi_{i-1,2} - \left[-\Delta y_{1_i} + \Delta y_{1_{i-1}}\left(1 + \frac{(ds/dx)_i}{(ds/dx)_{i-1}}\right) - \delta_{1_{i-1}}\left(1 + \frac{(ds/dx)_i}{(ds/dx)_{i-1}}\right)\right]\frac{U_{i-1}}{(ds/dx)_{i-1}}$$

$$D_4 = -(\psi_{i-1,2} + \psi_{i+1,2})$$

$$D_5 = -(\psi_{i-1,3} + \psi_{i+1,3})$$

$$\vdots$$

$$D_{k\max} = -(\psi_{i-1,j\max-1} + \psi_{i+1,j\max-1} + \psi_{i,j\max})$$

The coefficients a and b which appear in the integral boundary-layer equations [Eqs. (4) and (5)] are

$$a_{11} = \frac{2}{H_{12}} \qquad a_{21} = 1$$

$$a_{12} = \frac{-2\delta_1}{H_{12}^2} \qquad a_{22} = \delta_1 \left[\frac{1}{H_{32}} \frac{\mathrm{d}H_{32}}{\mathrm{d}H_{12}} - \frac{1}{H_{12}} \right]$$

$$b_{11} = -2\left(\frac{2+H_{12}}{H_{12}}\right)\frac{\delta_1}{U} \quad b_{21} = -\frac{3\delta_1}{U}$$

$$b_{12} = C_f \qquad b_{22} = \left(\frac{H_{12}}{H_{32}}\right)\left(\frac{2\nu}{U}\right)\int_0^\infty \left[\frac{\partial(u/U)}{\partial y}\right]^2 \mathrm{d}y$$

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